

Uniform bounds on locations of zeros of partial theta function

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Abstract

We consider the partial theta function $\theta(q, z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, where $(q, z) \in \mathbb{C}^2$, $|q| < 1$. We show that for any $0 < \delta_0 < \delta < 1$, there exists $n_0 \in \mathbb{N}$ such that for any q with $\delta_0 \leq |q| \leq \delta$ and for any $n \geq n_0$ the function θ has exactly n zeros with modulus $< |q|^{-n-1/2}$ counted with multiplicity.

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1 Introduction

We consider the bivariate series $\theta(q, z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, where $(q, z) \in \mathbb{C}^2$, $|q| < 1$. This series defines a *partial theta function*. The terminology is explained by the fact that the Jacobi theta function is defined by the series $\sum_{j=-\infty}^{\infty} q^{j^2} z^j$ and the following equality holds true: $\theta(q^2, z/q) = \sum_{j=0}^{\infty} q^{j^2} z^j$. The word “partial” is justified by the summation in θ ranging from 0 to ∞ and not from $-\infty$ to ∞ . In what follows we consider z as a variable and q as a parameter. For each fixed value of the parameter q the function θ is an entire function in the variable z .

The function θ finds applications in various domains, such as statistical physics and combinatorics (see [18]), Ramanujan type q -series (see [19]), the theory of (mock) modular forms (see [3]), asymptotic analysis (see [2]), and also in problems concerning real polynomials in one variable with all roots real (such polynomials are called *hyperbolic*, see [4], [5], [16], [15], [6], [14] and [7]). Other facts about θ can be found in [1].

The zeros of θ depend on the parameter q . For some values of q (called *spectral*) confluence of zeros occurs, so it would be correct to regard the zeros as multivalued functions of q ; about the spectrum of θ see [14], [12] and [13].

Notation 1. We denote by \mathbb{D}_ρ the open disk in the q -space centered at 0 and of radius ρ , by \mathcal{C}_ρ the corresponding circumference, and by $A_{\delta_0, \delta}$ the closed annulus $\{q \in \mathbb{C} \mid \delta_0 \leq |q| \leq \delta\}$.

In the present paper we prove the following theorem:

Theorem 2. *For any couple of numbers (δ_0, δ) such that $0 < \delta_0 < \delta < 1$, there exists $n_0 \in \mathbb{N}$ such that for any $q \in A_{\delta_0, \delta}$ and for any $n \geq n_0$ the function θ has exactly n zeros in $\mathbb{D}_{|q|^{-n-1/2}}$ counted with multiplicity.*

Remarks 3. (1) The proof of the theorem is based on a comparison between θ and the function

$$u(q, z) := \prod_{\nu=1}^{\infty} (1 + q^\nu z) \quad (1)$$

We use the equality

$$u = \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j / (q; q)_j, \quad (2)$$

where $(q; q)_j := (1 - q)(1 - q^2) \cdots (1 - q^j)$ is the q -Pochhammer symbol; it follows directly from Problem I-50 of [17] (see pages 9 and 186 of [17]). The analog of the above theorem for the *deformed exponential function* $\sum_{j=0}^{\infty} q^{j(j+1)/2} z^j / j!$ is proved in a non-published text by A. E. Eremenko using a different method.

(2) For q close to 0 the zeros of θ are of the form $-q^{-\ell}(1 + o(1))$, $\ell \in \mathbb{N}$, see more details about this in [9], [10] and [11].

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2 Proofs

Proof of Theorem 2. It is shown in [9] that for $0 < |q| \leq 0.108$ the zeros of θ can be expanded in convergent Laurent series. Recall that the function u (defined by (1)) satisfies equality (2), i.e. the zeros of u are the numbers $-q^{-\ell}$, $\ell \in \mathbb{N}$. We show that for $n \in \mathbb{N}$ sufficiently large the functions u and θ have one and the same number of zeros in the open disk $\mathbb{D}_{|q|^{-n-1/2}}$. To this end we show that for the restrictions u^0 and θ^0 of u and θ to the circumference $\mathcal{C}_{|q|^{-n-1/2}}$ one has $|u^0 - \theta^0 / (q; q)_n| < |u^0|$ after which we apply the Rouché theorem.

For $0 < |q| \leq 0.108$ one can establish a bijection between the zeros of θ and u , because their ℓ th zeros are of the form $-q^{-\ell}(1 + o(1))$ and the moduli of the zeros increase with ℓ , see part (2) of Remarks 3.

Set $P_k(|q|) := \prod_{\ell=0}^k (1 - |q|^{\ell+1/2})$, $k \in \mathbb{N} \cup \infty$. For $|u^0|$ one obtains the estimation

$$|u^0| \geq |q|^{-n^2/2} P_{n-1}(|q|) P_{\infty}(|q|) > |q|^{-n^2/2} (P_{\infty}(|q|))^2 \geq |q|^{-n^2/2} (P_{\infty}(\delta))^2. \quad (3)$$

Indeed, for $|z| = |q|^{-n-1/2}$ one can set $z := |q|^{-n-1/2} \omega$, $|\omega| = 1$. For $1 \leq \nu \leq n$ (resp. for $\nu > n$), the factor $(1 + q^{\nu} z)$ in (1) is of the form $(1 - |q|^{-\ell-1/2} \omega_{\ell})$, where $\ell = n - \nu$ and $|\omega_{\ell}| = 1$ (resp. of the form $(1 - |q|^{\ell+1/2} \omega_{\ell}^*)$, where $\ell = \nu - n - 1$ and $|\omega_{\ell}^*| = 1$). Thus

$$u(q, |q|^{-n-1/2} \omega^{-n-1/2}) = \prod_{\ell=0}^{n-1} (1 - |q|^{-\ell-1/2} \omega_{\ell}) \prod_{\ell=0}^{\infty} (1 - |q|^{\ell+1/2} \omega_{\ell}^*).$$

The first of the factors in the right-hand side can be represented in the form $|q|^{-n^2/2} \tilde{\omega} \prod_{\ell=0}^{n-1} (1 - |q|^{\ell+1/2} \omega_{\ell}^{**})$ with $|\tilde{\omega}| = |\omega_{\ell}^{**}| = 1$. Therefore

$$u(q, |q|^{-n-1/2} \omega^{-n-1/2}) = |q|^{-n^2/2} \tilde{\omega} \prod_{\ell=0}^{n-1} (1 - |q|^{\ell+1/2} \omega_{\ell}^{**}) \prod_{\ell=0}^{\infty} (1 - |q|^{\ell+1/2} \omega_{\ell}^*).$$

The modulus of the right-hand side is minimal for $\omega_{\ell}^* = \omega_{\ell}^{**} = 1$ in which case one obtains the leftmost inequality in (3).

Consider the monomial $\beta_j := \alpha_j z^j$ in the series $u - \theta / (q; q)_n$. Hence for $j = n$ it vanishes and for $j > n$ one has

$$\alpha_j = q^{j(j+1)/2}(1/(q; q)_j - 1/(q; q)_n) = q^{j(j+1)/2}U_{j,n} \quad , \quad \text{where}$$

$$U_{j,n} := (1 - \prod_{\ell=n+1}^j (1 - q^\ell))/(q; q)_j \quad ,$$

so for $|z| = |q|^{-n-1/2}$ one has $|\beta_j| = |q|^{-n^2/2+(j-n)^2/2}|U_{j,n}|$. One can observe that $U_{j,n} = q^{n+1} + O(q^{n+2})$. Set

$$U_{j,n} := \sum_{\nu \geq n+1} u_{j,n;\nu} q^\nu \quad \text{and} \quad U := ((\prod_{\ell=1}^{\infty} (1 + q^\ell)) - 1)/(q; q)_\infty = \sum_{\nu=1}^{\infty} u_\nu q^\nu \quad .$$

The Taylor series of U converges for $|q| < 1$ because the infinite products defining U converge. Clearly $u_{j,n;\nu} \in \mathbb{Z}$, $u_\nu \in \mathbb{N}$ (because all coefficients of the series $1/(q; q)_j$ and $1/(q; q)_\infty$ are positive integers) and $u_{j,n;n+1} = u_1 = 1$.

The following lemma explains in what sense the series U majorizes the series $U_{j,n}$.

Lemma 4. *One has $|u_{j,n;n+\nu}| \leq u_\nu$, $\nu \in \mathbb{N}$.*

Before proving Lemma 4 (the proof is given at the end of the paper) we continue the proof of Theorem 2.

Set $R(|q|) := \sum_{j>n} |q|^{(j-n)^2/2}$. The following inequality results immediately from the lemma:

$$Z_1 := \sum_{j>n} |\beta_j| \leq |q|^{-n^2/2} |q|^n U(|q|) R(|q|) \leq |q|^{-n^2/2} \delta^n U(\delta) R(\delta) \quad . \quad (4)$$

The first condition which we impose on the choice of n is the following inequality to be fulfilled:

$$\delta^n U(\delta) R(\delta) < (P_\infty(\delta))^2/4 \quad . \quad (5)$$

For $j < n$ and $|z| = |q|^{-n-1/2}$ one has $|\beta_j| = |q|^{-n^2/2+(j-n)^2/2} |\tilde{U}_{j,n}|$, where

$$\tilde{U}_{j,n} := (\prod_{\ell=j+1}^n (1 - q^\ell) - 1)/(q; q)_n \quad . \quad (6)$$

Hence $|\tilde{U}_{j,n}| \leq T(|q|) := (\prod_{\ell=1}^{\infty} (1 + |q|^\ell) + 1)/(|q|; |q|)_\infty$ and

$$|\beta_j| \leq |q|^{-n^2/2} |q|^{(j-n)^2/2} T(\delta) \quad (7)$$

Choose $m \in \mathbb{N}$ such that $T(\delta) \sum_{s=m}^{\infty} \delta^{s^2/2} \leq (P_\infty(\delta))^2/4$. Inequality (7) implies that

$$Z_2 := \sum_{j=0}^{n-m} |\beta_j| \leq |q|^{-n^2/2} (P_\infty(\delta))^2/4 \quad (8)$$

Notice that for $n < m$ the above sum is empty and the inequality trivially holds true.

The finite sum

$$Z_3 := \sum_{j=n-m+1}^{n-1} |\beta_j| \quad (9)$$

is of the form $|q|^{-n^2/2} O(|q|^n)$. Indeed, consider formula (6). There exists $M > 0$ depending only on δ_0 and δ such that

$$0 < |1/(q; q)_n| \leq 1/(|q|; |q|)_n < 1/(|q|; |q|)_\infty \leq M \quad \text{for} \quad \delta_0 \leq |q| \leq \delta \quad .$$

Thus

$$|\tilde{U}_{j,n}| \leq M \left(\prod_{\ell=j+1}^n (1 + |q|^\ell) - 1 \right) .$$

The index j can take only the values $n - m + 1, \dots, n - 1$. In the last product each monomial $|q|^\ell$ can be represented in the form $|q|^n |q|^{\ell-n}$, where $\ell - n = 2 - m, \dots, 0$. The modulus of each factor $|q|^{\ell-n}$ is not larger than $1/\delta_0^{\max(0, m-2)}$. Therefore

$$|\tilde{U}_{j,n}| \leq M((1 + |q|^n/\delta_0^{\max(0, m-2)})^{m-1} - 1) = O(|q|^n) .$$

The sum Z_3 (see (9)) can be made less than $|q|^{-n^2/2}(P_\infty(\delta))^2/4$ by choosing n large enough. Thus inequalities (3), (4) and (8) yield

$$|u^0 - \theta^0/(q; q)_n| \leq Z_1 + Z_2 + Z_3 \leq (3/4)|q|^{-n^2/2}(P_\infty(\delta))^2 < |q|^{-n^2/2}(P_\infty(\delta))^2 \leq |u^0|$$

which proves the theorem. \square

Proof of Lemma 4. We first compare the coefficients of the series

$$\prod_{\ell=p}^r (1 + q^\ell) - 1 = \sum_{\nu \geq p} \gamma_\nu^1 q^\nu \quad \text{and} \quad \prod_{\ell=p}^r (1 - q^\ell) - 1 = \sum_{\nu \geq p} \gamma_\nu^2 q^\nu \quad , \quad p \leq r .$$

They are obtained respectively as a sum of the non-negative coefficients of monomials and as a linear combination of the same coefficients some of which are taken with the $+$ and the rest with the $-$ sign. Therefore $\gamma_\nu^1 \geq |\gamma_\nu^2|$, $\nu \geq p$. This means that $|u_{j,n;\nu}| \leq v_{j,n;\nu} \leq v_{\infty,n;\nu}$, where

$$V_{j,n} := \left(\prod_{\ell=n+1}^j (1 + q^\ell) - 1 \right) / (q; q)_j = \sum_{\nu \geq n+1} v_{j,n;\nu} q^\nu \quad , \quad V_{\infty,0} = U \quad \text{and} \quad v_{\infty,0;\nu} = u_\nu .$$

To prove the lemma it suffices to show that

$$v_{\infty,n;n+\nu} \leq v_{\infty,0;\nu} . \tag{10}$$

Consider the series $S_r := \prod_{\ell=r+1}^\infty (1 + q^\ell) - 1 = \sum_{\nu \geq r+1} s_{r;\nu} q^\nu$ for $r = 0$ and $r = n$. Compare the coefficients $s_{0;\nu}$ and $s_{n;n+\nu}$. The coefficient $s_{0;\nu}$ is equal to the number of ways in which ν can be represented as a sum of distinct natural numbers forming an increasing sequence whereas $s_{n;n+\nu}$ is the number of ways in which $n + \nu$ can be represented as a sum of distinct natural numbers $\geq n + 1$ forming an increasing sequence. Clearly $s_{n;n+\nu} \leq s_{0;\nu}$. This implies inequality (10) and the lemma, because one has $V_{\infty,r} = S_r/(q; q)_\infty$ and the coefficients of the series $1/(q; q)_\infty$ are all positive. \square

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